

A Simple Stationary Dynamo Model *

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(Z. Naturforsch. 27 a, 1350—1354 [1972] ; received 26 June 1972)

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Solutions of the stationary dynamo equations are derived such that outside a torus the magnetic field is the axisymmetric vacuum field of a circular loop, while inside the torus in the limit of large aspect ratio both the velocity and the magnetic fields have helical symmetry.

In 1958 HERZENBERG¹ and BACKUS² have showed that motions in an electrically conducting fluid sphere can generate magnetic fields, thus demonstrating that the “dynamo mechanism” can explain the origin of stellar and terrestrial magnetic fields. Since then a lot of papers concerned with this “dynamo problem” have been published. A relatively complete reference list can be found in Weiss’s review article³. One of the theoretically interesting aspects of the dynamo problem is Cowling’s theorem (see, for instance, BRAGINSKII⁴) stating that a dynamo is impossible if both the velocity and magnetic fields are axially symmetric. On the other hand, the magnetic field at the surface of the earth possesses a high degree of axial symmetry. Usually, it is hoped that the field deviation from axisymmetry at the surface of the earth can in some way or other be related to the structure of the velocity and magnetic fields in the earth’s interior. The following model shows that such a connection is doubtful because here the field outside the conductor is exactly axisymmetric.

Let us write the stationary “dynamo equations” for an incompressible fluid in the form

$$\eta \operatorname{curl} \mathbf{B} + \nabla \Phi - \mathbf{v} \times \mathbf{B} = 0, \quad (1)$$

$$\operatorname{div} \mathbf{B} = 0, \quad (2)$$

$$\operatorname{div} \mathbf{v} = 0 \quad (3)$$

where \mathbf{v} is the velocity, \mathbf{B} the magnetic field, and Φ the electric potential. The magnetic diffusivity η is assumed to be constant in the interior of a singly connected region and infinite outside.

Introduce cylindrical coordinates $\varrho, \vartheta, \zeta$ and orthogonal toroidal coordinates s, φ, z by (Fig. 1)

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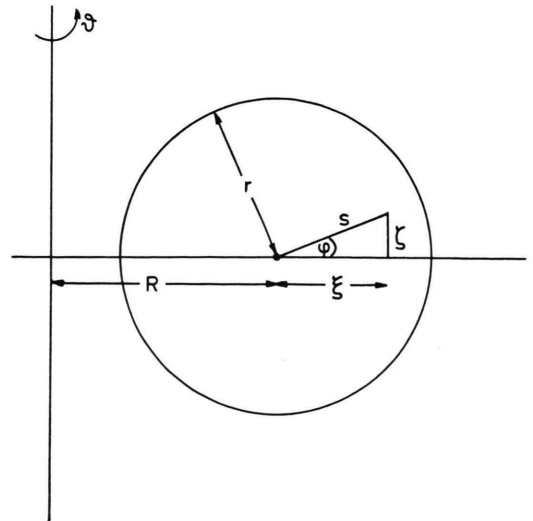


Fig. 1. Toroidal coordinates s, φ, z where $z = R \vartheta$.

$$\varrho = R + \xi, \quad \xi = s \cos \varphi,$$

$$\vartheta = R^{-1} z,$$

$$\zeta = s \sin \varphi.$$

This yields the metrical fundamental form

$$\begin{aligned} dl^2 &= d\varrho^2 + \varrho^2 d\vartheta^2 + d\zeta^2 = \\ &= ds^2 + s^2 d\varphi^2 + (1 + R^{-1} \varrho \cos \varphi)^2 dz^2. \end{aligned}$$

Let r be the minor radius of an axisymmetric torus with circular meridional cut. Then the interior of the torus is described by

$$0 \leq s \leq r < R, \quad 0 \leq \vartheta < 2\pi.$$

Introducing the natural components of \mathbf{B} and \mathbf{v} by

$$\mathbf{B} = \begin{pmatrix} B_s \\ B_\varphi \\ B_z \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_s \\ v_\varphi \\ v_z \end{pmatrix}$$

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we can write the vector operations occurring in (1), (2) as

$$\operatorname{div} \mathbf{B} = \frac{1}{s e_z} \left(\frac{1}{s} \frac{\partial}{\partial s} s e_z B_s + \frac{1}{s} \frac{\partial}{\partial \varphi} e_z B_\varphi + \frac{\partial}{\partial z} B_z \right) \quad (4)$$

$$\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial s} \\ \frac{1}{s} \frac{\partial \Phi}{\partial \varphi} \\ \frac{1}{e_z} \frac{\partial \Phi}{\partial z} \end{pmatrix}, \quad (5)$$

$$\operatorname{curl} \mathbf{B} = \begin{pmatrix} \frac{1}{s e_z} \left(\frac{\partial}{\partial \varphi} e_z B_z - \frac{\partial}{\partial z} s B_\varphi \right) \\ \frac{1}{e_z} \left(\frac{\partial}{\partial z} B_s - \frac{\partial}{\partial s} e_z B_z \right) \\ \frac{1}{s} \left(\frac{\partial}{\partial s} s B_\varphi - \frac{\partial}{\partial \varphi} B_s \right) \end{pmatrix} \quad (6)$$

where

$$e_z = 1 + \varepsilon(s/r) \cos \varphi \quad (7)$$

and

$$\varepsilon = (r/R) < 1$$

is the inverse aspect ratio of the torus.

Let us look for solutions of Eqs. (1)–(3) with the following properties: the velocity \mathbf{v} and the current density $\operatorname{curl} \mathbf{B}$ are non-zero only in the interior of the torus and their normal components vanish on the surface, while the \mathbf{B} -field is continuous at the surface and outside the torus \mathbf{B} is the poloidal vacuum field of a circular loop at $s = 0$. Such a configuration is unaltered if the torus is embedded in a sphere of constant conductivity.

The field of the circular loop is well-known⁵. The components of its vector potential in cylindrical coordinates are

$$A_\varphi = A_\zeta = 0,$$

$$A_\theta = A \int_0^\pi \frac{R \cos \vartheta d\vartheta}{(R^2 + \varrho^2 + \zeta^2 - 2R\varrho \cos \vartheta)^{1/2}} \\ = 4 \frac{A}{\kappa} \left(\frac{R}{\varrho} \right)^{1/2} \left[\left(1 - \frac{\kappa^2}{2} \right) K - E \right] \quad (8)$$

where

$$\kappa^2 = \frac{4R\varrho}{(R+\varrho)^2 + \zeta^2} = \frac{1 + (\xi/R)}{1 + (\xi/R) + (s^2/4R^2)}, \quad (9)$$

$$K(\kappa) = \int_0^{\pi/2} (1 - \kappa^2 \cos^2 \vartheta)^{-1/2} d\vartheta,$$

$$E(\kappa) = \int_0^{\pi/2} (1 - \kappa^2 \cos^2 \vartheta)^{1/2} d\vartheta$$

are the complete elliptic integrals of the first and second kinds, and the constant A is proportional to the current in the loop.

The representation (8) yields the field components⁵

$$\left. \begin{aligned} B_\varrho &= 2A \frac{\zeta}{R+\xi} [(2R+\xi)^2 + \zeta^2]^{-1/2} \\ &\quad \left(-K + \frac{2R^2 + 2R\xi + s^2}{s^2} E \right) \\ B_\theta &= 0 \\ B_\zeta &= 2A [(2R+\xi)^2 + \zeta^2]^{-1/2} \\ &\quad \left(K - \frac{2R\xi + s^2}{s^2} E \right). \end{aligned} \right\} \quad (10)$$

We now consider the field (10) at the surface $s = r$ of the torus for the case of small inverse aspect ratio $\varepsilon \ll 1$. It is seen from Eq. (9) that

$$\kappa^2 = 1 - (s^2/4r^2) \varepsilon^2 + O(\varepsilon^3).$$

Here a difficulty arises from the fact that K and E have an essential singularity at $\kappa = 1$. These functions can nevertheless be represented as convergent series in

$$\kappa'^2 = 1 - \kappa^2, \quad 0 < \kappa' < 1$$

when we admit that the coefficients are slowly varying functions of κ' (see Ref. 6 and 7)

$$K = A + \frac{1}{4} (A-1) \kappa'^2 + \frac{9}{64} (A - \frac{7}{6}) \kappa'^4 + \dots,$$

$$E = 1 + \frac{1}{2} (A - \frac{1}{2}) \kappa'^2 + \frac{3}{16} (A - \frac{13}{12}) \kappa'^4 + \dots,$$

$$A = \ln(4/\kappa').$$

The field (10) then yields for $s = r$

$$B_\varrho = -8A r^{-1} \sin \varphi [1 - \frac{1}{2} \varepsilon \cos \varphi + O(\varepsilon^2)],$$

$$B_\theta = 0,$$

$$B_\zeta = -8A r^{-1} [\cos \varphi - \frac{1}{2} \varepsilon \lambda + \frac{1}{2} \varepsilon \sin^2 \varphi + O(\varepsilon^2)],$$

$$\lambda = \ln(8/\varepsilon),$$

and the components in the toroidal coordinate system are

$$\left. \begin{aligned} B_s &= B_\varrho \cos \varphi - B_\zeta \sin \varphi \\ &= -4r^{-1} A \varepsilon (\lambda - 1) \sin \varphi + O(\varepsilon^2), \\ B_\varphi &= B_\varrho \sin \varphi + B_\zeta \cos \varphi \\ &= -8r^{-1} A [1 - \frac{1}{2} \varepsilon \lambda \cos \varphi + O(\varepsilon^2)] \\ B_z &= 0. \end{aligned} \right\} \quad (11)$$

In the interior of the torus we also try to solve Eqs. (1)–(3) as power series in ε

$$\left. \begin{aligned} \Phi &= \Phi_0 + \varepsilon \Phi_1 + \dots, \\ \mathbf{B} &= \mathbf{B}_0 + \varepsilon \mathbf{B}_1 + \dots, \\ \mathbf{v} &= \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \dots. \end{aligned} \right\} \quad (12)$$

However, we have to admit that because of the boundary conditions (11) the coefficients in the

series (12) weakly depend on ε . Up to linear terms in ε it is found from Eqs. (1)–(3), (4)–(7) that

$$\left. \begin{aligned} \eta \operatorname{curl}_0 \mathbf{B}_0 + \nabla_0 \Phi_0 - \mathbf{v}_0 \times \mathbf{B}_0 &= 0, \\ \operatorname{div}_0 \mathbf{B}_0 &= 0, \\ \operatorname{div}_0 \mathbf{v}_0 &= 0, \end{aligned} \right\} \quad (13)$$

$$\nabla_0 \Phi_1 - \mathbf{D} - \mathbf{v}_1 \times \mathbf{B}_0 = 0, \quad (14)$$

$$\mathbf{D} = \mathbf{v}_0 \times \mathbf{B}_1 - \eta \operatorname{curl}_0 \mathbf{B}_1 - \eta \operatorname{curl}_1 \mathbf{B}_0 - \nabla_1 \Phi_0,$$

$$\operatorname{div}_0 \mathbf{B}_1 + \operatorname{div}_1 \mathbf{B}_0 = 0, \quad (15)$$

$$\operatorname{div}_0 \mathbf{v}_1 + \operatorname{div}_1 \mathbf{v}_0 = 0 \quad (16)$$

where in the system s, φ, z

$$\operatorname{div}_0 \mathbf{B}_i = \frac{1}{s} \frac{\partial}{\partial s} s B_{is} + \frac{1}{s} \frac{\partial}{\partial \varphi} B_{i\varphi} + \frac{\partial}{\partial z} B_{iz},$$

$$i = 0, 1,$$

$$\operatorname{div}_1 \mathbf{B}_0 = \frac{1}{sr} \left(\frac{\partial}{\partial s} s^2 B_{0s} \cos \varphi + s \frac{\partial}{\partial \varphi} B_{0\varphi} \cos \varphi \right),$$

(analogously for \mathbf{v})

$$\nabla_0 \Phi_i = \begin{pmatrix} \frac{\partial}{\partial s} \Phi_i \\ \frac{1}{s} \frac{\partial}{\partial \varphi} \Phi_i \\ \frac{\partial}{\partial z} \Phi_i \end{pmatrix}, \quad i = 0, 1,$$

$$\nabla_1 \Phi_0 = -\frac{s}{r} \cos \varphi \frac{\partial \Phi_0}{\partial z} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\operatorname{curl}_0 \mathbf{B}_i = \begin{pmatrix} \frac{1}{s} \frac{\partial}{\partial \varphi} B_{iz} - \frac{\partial}{\partial z} B_{i\varphi} \\ \frac{\partial}{\partial z} B_{is} - \frac{\partial}{\partial s} B_{iz} \\ \frac{1}{s} \frac{\partial}{\partial s} s B_{i\varphi} - \frac{1}{s} \frac{\partial}{\partial \varphi} B_{is} \end{pmatrix},$$

$$i = 0, 1,$$

$$\operatorname{curl}_1 \mathbf{B}_0 = \begin{pmatrix} \frac{s}{r} \frac{\partial}{\partial z} B_{0\varphi} \cos \varphi + \frac{1}{r} B_{0z} \sin \varphi \\ -\frac{s}{r} \frac{\partial}{\partial z} B_{0s} \cos \varphi - \frac{1}{r} B_{0z} \cos \varphi \\ 0 \end{pmatrix}.$$

Equations (13) have been solved in Ref. 8 for helical symmetry. The solution can be written in the form

$$\begin{aligned} B_{0s} &= 0, \\ B_{0\varphi} &= q(y - fks), \\ B_{0z} &= q(yks + f), \\ v_{0s} &= -\frac{1}{s} \frac{\partial G}{\partial u}, \end{aligned}$$

$$\begin{aligned} v_{0\varphi} &= q \left(\frac{\partial G}{\partial s} - gks \right), \\ v_{0z} &= q \left(\frac{\partial G}{\partial s} ks + g \right), \\ \Phi_0 &= \Psi_1 \sin u + \Psi_2 \sin 2u, \\ f &= f_0 + f_1 \cos u, \\ G &= G_1 \sin u, \\ g &= g_1 \sin u + g_2 \sin 2u. \end{aligned}$$

Here

$$\begin{aligned} u &= \varphi + kz, \\ q &= (1 + k^2 s^2)^{-1}, \end{aligned}$$

$2\pi/k$ is the period in the z -direction, and $f_0, f_1, \Psi_1, \Psi_2, G_1, g_1, g_2$ are functions of s which can be expressed by derivatives of $y(s)$

$$\left. \begin{aligned} f_0 &= \frac{(sqy)'}{2kq^2s}, \quad f_1 = \left(-\frac{yf_0'}{kq} \right)^{1/2}, \\ G_1 &= 2\eta y^{-1} ksqf_1, \quad \Psi_1 = q(f_0 G_1 + \eta s f_1'), \\ \Psi_2 &= \frac{1}{4} q f_1 G_1, \\ g_2 &= (4qy)^{-1} [qf_1 G_1' - (qf_1)' G_1], \\ g_1 &= -(qy)^{-1} [\eta (sqf_1')' - \eta s^{-1} f_1 + (f_0 q)' G_1]. \end{aligned} \right\} \quad (17)$$

Apart from an additive constant in f_0 , the first of the Eqs. (17) is satisfied if we put

$$y = s \int_r^s k f_0'(t) q(t) dt - (1/s) \int_0^s k f_0'(t) q(t) t^2 dt.$$

This expression shows that y and kf_0' have opposite sign (thus f_1 is real) if we choose for kf_0' a function which does not vanish in the interval $0 < s < r$. At the points $s = 0, s = r$ let us impose the following conditions

$$\begin{aligned} kf_0' &\sim O(s), \quad s \rightarrow 0, \\ kf_0' &\sim O[(s-r)^4], \quad s \rightarrow r. \end{aligned} \quad (18)$$

With the boundary condition

$$\mathbf{B}_0 = -\frac{8A}{r} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

we then find for $s \rightarrow r$:

$$\begin{aligned} y &= -\frac{8A}{r} \left(1 - \frac{s-r}{r} \right) + O[(s-r)^2], \\ f_0 &= 8Ak + O[(s-r)^5], \\ f_1 &\sim G_1 \sim O[(s-r)^2], \\ \Psi_1 &\sim O[s-r], \\ \Psi_2 &\sim O[(s-r)^4], \\ g_2 &\sim O[(s-r)^3], \\ g_1 &\sim O[(s-r)^0]. \end{aligned}$$

So the normal component of the velocity and the current density as well as the tangential component of the electric field vanish at the surface of the torus.

Relation (18) gives for $s \rightarrow 0$:

$$\begin{aligned} y &= -Cks + O(s^3), \\ f_0 &= -C + O(s^2), \\ f_1 &\sim G_1 \sim g_1 \sim \Psi_1 \sim s, \\ g_2 &\sim \Psi_2 \sim s^2, \end{aligned}$$

where

$$C = \int_0^r f_0'(t) q(t) dt.$$

Hence it has been shown that fields are regular and, in addition, that

$$B_0^2 = q(y^2 + f^2) \neq 0 \quad \text{for } 0 \leq s \leq r. \quad (19)$$

A further requirement is the 2π -periodicity in ϑ , which implies that

$$m = kR = (kr/\varepsilon)$$

is an integer. Thus, if kr is $O(\varepsilon^0)$, m must be $O(\varepsilon^{-1})$.

Once the zero-order fields are known we ask whether the linear inhomogeneous Eqs. (14)–(16) for \mathbf{v}_1 , \mathbf{B}_1 , Φ_1 together with the boundary conditions

$$\mathbf{B}_1 = \frac{4A}{r} \begin{pmatrix} -(\lambda - 1) \sin \varphi \\ \lambda \cos \varphi \\ 0 \end{pmatrix}, \quad v_{1s} = 0 \quad \text{for } s = r \quad (20)$$

are solvable.

First we observe that \mathbf{v}_1 can be eliminated from Eq. (14) by forming the scalar product with \mathbf{B}_0

$$\mathbf{B}_0 \cdot \nabla_0 \Phi_1 = \mathbf{B}_0 \cdot \mathbf{D}. \quad (21)$$

Furthermore, the component \mathbf{v}_1^\perp of \mathbf{v}_1 which is perpendicular to \mathbf{B}_0 can be computed algebraically from Eq. (14) if Eq. (21) is satisfied.

$$\mathbf{v}_1^\perp = B_0^{-2} \mathbf{B}_0 \times (\nabla_0 \Phi_1 - \mathbf{D}).$$

Here B_0^{-2} is finite for $0 \leq s \leq r$ because of condition (19). Finally, if we introduce the parallel component of \mathbf{v}_1 by

$$\mathbf{v}_1 = \alpha \mathbf{B}_0 + \mathbf{v}_1^\perp$$

then Eq. (16) is equivalent to

$$\mathbf{B}_0 \cdot \nabla_0 \alpha = -\operatorname{div}_0 \mathbf{v}_1^\perp - \operatorname{div}_0 \mathbf{v}_0. \quad (22)$$

Equations (21) and (22) have the same structure. For $s \neq 0$ the right-hand side of Eq. (21) as well as Φ_1 can be considered as functions of s , φ , and u . Since

$$\mathbf{B}_0 \cdot \nabla_0 \Phi_1 = \frac{1}{s} B_{0\varphi} \frac{\partial \Phi_1}{\partial \varphi} + \frac{y}{s} \frac{\partial \Phi_1}{\partial u},$$

Eq. (21) can be written in the form

$$a(u, s) \frac{\partial \Phi_1}{\partial \varphi} + \frac{\partial \Phi_1}{\partial u} = h_1(s, u) \cos \varphi + h_2(s, u) \sin \varphi$$

where

$$a = y^{-1} B_{0\varphi}.$$

If we put

$$\Phi_1 = \chi_1(s, u) \cos \varphi + \chi_2(s, u) \sin \varphi$$

then χ_1 , χ_2 must satisfy

$$\begin{aligned} a \chi_2 + (\partial \chi_1 / \partial u) &= h_1, \\ -a \chi_1 + (\partial \chi_2 / \partial u) &= h_2, \end{aligned}$$

or in the complex notation

$$\begin{aligned} h &= h_1 + i h_2, \\ \chi &= \chi_1 + i \chi_2, \\ (\partial \chi / \partial u) - i a \chi &= h. \end{aligned} \quad (23)$$

The general solution of Eq. (23) is

$$\begin{aligned} \chi(s, u) &= \exp[i \int_0^u a(s, t) dt] \{ \chi_0(s) + \int_0^u h(s, t) \\ &\quad \cdot \exp[-i \int_0^t a(s, t') dt'] \}. \end{aligned}$$

Here the function $\chi_0(s)$ is determined by the requirement that χ be periodic in u with the period 2π

$$\chi(s, 2\pi) = \chi(s, 0)$$

yielding

$$\begin{aligned} \chi_0 &= 2\pi [1 - \exp(2\pi i \bar{a})]^{-1} \\ &\quad \cdot \exp(2\pi i \bar{a}) h \exp[-i \int_0^u a(s, t) dt] \end{aligned} \quad (24)$$

where the bar

$$\bar{\cdots} = (1/2\pi) \int_0^{2\pi} \cdots du$$

indicates the u -average along $s = \text{const}$, $\varphi = \text{const}$. The expression (24) shows that χ_0 , and hence Φ_1 and α , are finite if $\bar{a} \neq 0$. From

$$\overline{B_{0\varphi}} = q(y - f_0 k s) = q \left[y - \frac{(sqy)'}{2q^2} \right] = -\frac{s^2}{2} \left(\frac{y}{s} \right)'$$

and

$$\left(\frac{y}{s} \right)' = \frac{2}{s^3} \int_0^s k f_0'(t) q(t) t^2 dt \neq 0 \quad \text{for } 0 < s \leq r$$

it is found that

$$\bar{a} = -\frac{s^2}{2y} \left(\frac{y}{s} \right)'$$

does not vanish for $0 < s \leq r$, and for $s \rightarrow 0$ tends to

$$\bar{a} \sim O(s^2).$$

The rest of the problem thus consists in checking whether a solution of Eq. (15) exists such that at $s = 0$ all fields are finite and at $s = r$ the boundary conditions (20) are satisfied. The form of Eq. (15) suggests putting

$$\mathbf{B}_1 = \mathbf{b} \cos \varphi + \mathbf{d} \sin \varphi$$

where $b_s, b_\varphi, d_s, d_\varphi$ depend on s alone and b_z, d_z depend on s and u . Then

$$\left. \begin{aligned} (s b_s)' + d_\varphi &= 0, \\ \frac{1}{s} (s d_s)' - \frac{1}{s} b_\varphi &= \frac{q}{r} (y - f_0 k s), \\ b_z &= \bar{b}_z(s) - q r^{-1} f_1 s \cos u, \\ d_z &= \bar{d}_z(s) + q r^{-1} f_1 s \sin u. \end{aligned} \right\} \quad (25)$$

Let us consider the case $s \rightarrow 0$ and assume

$$\begin{aligned} b_s &= b_s^* s^4 + O(s^5), \\ b_\varphi &= b_\varphi^* s^4 + O(s^5), \\ d_s &= d_s^* s^4 + O(s^5), \\ d_\varphi &= d_\varphi^* s^4 + O(s^5), \\ b_z &\sim O(s^2), \\ d_z &\sim O(s^2), \\ \bar{b}_z &= b_z^* s^3 + O(s^4), \\ \bar{d}_z &= d_z^* s^3 + O(s^4), \end{aligned}$$

which is compatible with Equations (25). Then the right-hand side of Eq. (21) is

$$\mathbf{B}_0 \cdot \mathbf{D} \sim O(s^2)$$

and its u -average entering into formula (24) for $s \rightarrow 0$ is

$$\overline{\mathbf{B}_0 \cdot \mathbf{D}} \sim O(s^3).$$

However, choosing the coefficients $b_s^*, b_\varphi^*, d_s^*, d_\varphi^*$ results in

$$\mathbf{B}_0 \cdot \mathbf{D} \sim O(s^4).$$

This implies $\chi_0 \sim O(s^2)$ and, finally,

$$\Phi_1 \sim O(s^2).$$

The solution α of Eq. (22) must also be finite for $s \rightarrow 0$. Since

$$\overline{\text{div}_1 \mathbf{v}_0} = 0$$

the sufficient condition for α to be finite is

$$\overline{\text{div}_0 \mathbf{v}_1^\perp} \sim O(s^2). \quad (26)$$

For arbitrary coefficients b_z^*, d_z^* we have

$$\text{div}_0 \mathbf{v}_1^\perp \sim O(s).$$

However, since these coefficients to lowest order in s enter only into the term $\overline{\text{div}_0 B_0^{-2} \mathbf{B}_0 \times \text{curl}_0 \mathbf{B}_1}$, b_z^*, d_z^* can be determined such that condition (26) is satisfied.

Next, we consider the boundary conditions (20), which imply that for $s \rightarrow r$

$$\begin{aligned} d_s &= 4r^{-1} A (\lambda - 1) + O(s - r), \\ b_\varphi &= 4r^{-1} A \lambda + O(s - r), \\ b_s &\sim d_\varphi \sim b_z \sim d_z \sim O(s - r). \end{aligned}$$

Furthermore, it can easily be checked that for $s \rightarrow r$ the free functions contained in the solutions of Eqs. (25) can be chosen such that the vector \mathbf{D} is normal at the surface

$$D_\varphi \sim D_z \sim O(s - r).$$

Then it follows that

$$\Phi_1 \sim O(s - r).$$

Hence $\nabla_0 \Phi_1$ is normal and \mathbf{v}_1^\perp is tangential.

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